Hale School

Mathematics Specialist

Test 3 --- Term 2 2019

Vectors

/ 40

Name: Hector

Instructions:

- **Calculators are allowed**
- **1 page of external notes are allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

1. **[2, 4 = 6 marks]**

In the triangle OAB, $\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$.

(a) Determine $\angle AOB$

$$\cos \angle AOB = \frac{\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{3^2 + 4^2}\sqrt{1^2+2^2+2^2}}$$ ✓ formula

$$= -\frac{1}{3}$$

$$\angle AOB = 109.47^\circ \text{ or } 1.91$$ ✓ solution

(b) Determine $\overrightarrow{OP}$, where P is the point on AB such that OP is perpendicular to AB.

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$$

$$\overrightarrow{OP}: \underset{\sim}{r} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix}$$ ✓ defines $\overrightarrow{OP}$

$$\begin{pmatrix} 3-2\lambda \\ 2\lambda \\ 4-6\lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} = 0$$ $(\overrightarrow{OP} \perp \overrightarrow{AB})$ ✓ perpendicular

$$-6 + 4\lambda + 4\lambda - 24 + 36\lambda = 0$$

$$\lambda = \frac{15}{22}$$ ✓ determines sol^n for λ

$$\therefore \overrightarrow{OP} = \begin{pmatrix} \frac{18}{11} \\ \frac{15}{11} \\ -\frac{1}{11} \end{pmatrix}$$ ✓ substitutes correctly

2. **[2, 3 = 5 marks]**

Given the points A $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, B $\begin{pmatrix} -2 \\ 3 \\ -2 \end{pmatrix}$ and C $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$, determine:

(a) The equation of the line passing through A and B

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$$

✓ point ✓ dir$^{\underline{n}}$ vector

(b) The equation of the plane, Π , in normal form, passing through A, B and C.

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} -2 \\ -2 \\ 7 \end{pmatrix}$$

✓ determines dir$^{\underline{n}}$ vectors

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 16 \\ 26 \\ 12 \end{pmatrix} \text{ which is parallel to } \begin{pmatrix} 8 \\ 13 \\ 6 \end{pmatrix}$$

✓ normal

$$\underset{\sim}{r} \cdot \begin{pmatrix} 8 \\ 13 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} 8 \\ 13 \\ 6 \end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 8 \\ 13 \\ 6 \end{pmatrix} = 11$$

✓ eq$^{\underline{n}}$ of plane in normal form (simplified).

3. **[5 marks]**

A plane, Π , contains the line $\frac{2-x}{3}=\frac{y}{-4}=z+1$ and is parallel to $3\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$.

Find the cartesian equation of Π .

$\lambda = \frac{2-x}{3}$, $\lambda = \frac{y}{-4}$, $\lambda = z+1$

$x = 2-3\lambda$, $y = -4\lambda$, $z = \lambda - 1$ ✓ parametric

$\therefore$ dirn vector of line $\begin{pmatrix}-3\\-4\\1\end{pmatrix}$ ✓ determine dirn vector

Normal vector of Π : $\begin{pmatrix}-3\\-4\\1\end{pmatrix} \times \begin{pmatrix}3\\-2\\1\end{pmatrix} = \begin{pmatrix}-2\\6\\18\end{pmatrix} \, // \begin{pmatrix}-1\\3\\9\end{pmatrix}$

✓ cross product.

$\underset{\sim}{r} \cdot \begin{pmatrix}-1\\3\\9\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}\begin{pmatrix}-1\\3\\9\end{pmatrix}$ ✓ vector equation

$\begin{pmatrix}x\\y\\z\end{pmatrix}\begin{pmatrix}-1\\3\\9\end{pmatrix} = -11$

$-x+3y+9z = -11$ ✓ cartesian equation

4. **[5 marks]**

Determine the possible values of p and q if the system of equations

$$x - y + 2z = 1$$
$$2x - 5y + 5z = 9$$
$$3x + 3y + pz = q$$

has:

(a) a unique solution,

(b) no solution,

(c) infinite solutions.

Using CAS (ref[ ]) equations reduced to:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{1}{3} & -2\frac{1}{3} \\ 0 & 0 & p-4 & q+11 \end{array}\right]$$

✓✓ · method stated · reduced equations

a) $p \neq 4$ ✓

b) $p = 4,\ q \neq -11$ ✓

c) $p = 4,\ q = -11$ ✓

5. **[3, 4, 2 = 9 marks]**

A model aircraft follows a circuit in the plane defined by $\underset{\sim}{r} = (3\cos 4t)\underset{\sim}{i} - (2\sin 4t)\underset{\sim}{j}$.

(a) Determine the initial position of the aircraft and it's direction of motion.

$\underset{\sim}{r}(0) = 3\underset{\sim}{i}$

✓ initial

$\dot{\underset{\sim}{r}} = -12\sin 4t\,\underset{\sim}{i} - 8\cos 4t\,\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(0) = -8\underset{\sim}{j}$.

✓ dirⁿ determined

∴ Initial position $3\underset{\sim}{i}$ dirⁿ vertically downwards ✓ statement.

(b) Determine the time(s) throughout the flight where the velocity of the aircraft is parallel to $\underset{\sim}{i} + \underset{\sim}{j}$.

$$\begin{pmatrix} -12\sin 4t \\ -8\cos 4t \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

✓ parallel ~~stat~~ equated

$-12\sin 4t = \lambda$, $-8\cos 4t = \lambda$

✓ equates equating

$-12\sin 4t = -8\cos 4t$

$\tan 4t = \frac{8}{12}$

$4t = 0.588 + \pi n$

✓ determines one solⁿ

$t = 0.147 + \frac{\pi n}{4}$ $n \in \mathbb{Z}^+$

✓ generalises for n.

(c) Determine the distance travelled by the aircraft from $t = 1$ to $t = 2$.

$$\int_1^2 |\underset{\sim}{v}|\, dt$$

$$= \int_1^2 \sqrt{(-12\sin 4t)^2 + (-8\cos 4t)^2}\, dt$$

$= 10.41$ units .

6. **[4 marks]**

Given the lines $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 13 \\ -15 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

Determine whether the line intersect or are skew. If they intersect, find the point of intersection. If they are skew, show this clearly.

If they meet $\begin{pmatrix} 1-\lambda_1 \\ -2+3\lambda_1 \\ 3+2\lambda_1 \end{pmatrix} = \begin{pmatrix} 3-\lambda_2 \\ 13 \\ -15+4\lambda_2 \end{pmatrix}$ ✓ equate

$\underset{\sim}{j}$ comp: $\lambda_1 = 5$ ✓ resolves $\underset{\sim}{j}$

check $\underset{\sim}{i}$ + $\underset{\sim}{k}$

$\underset{\sim}{i}$: $1-\lambda_1 = 3-\lambda_2$
$1-5 = 3-\lambda_2$
$\lambda_2 = 7$

$\underset{\sim}{k}$: $3+2\lambda_1 = -15+4\lambda_2$
$3+2(5) = -15+4\lambda_2$
$\lambda_2 = 7$

✓ shows $\underset{\sim}{i}$ & $\underset{\sim}{k}$ equal

$\therefore$ lines intersect when $\lambda_1 = 5$ $\lambda_2 = 7$

i.e. Intersect at point $(-4, 13, 13)$ ✓ point.

7. **[6 marks]**

If the line $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is tangent to the sphere with equation

$\left| \underset{\sim}{r} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right| = a$, determine the value of a.

Subst line into plane :

$$\left| \begin{pmatrix} 1+2\lambda \\ 3+\lambda \\ -1+\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \right| = a$$ ✓ subst.

$$(2\lambda-2)^2 + (\lambda+3)^2 + (\lambda-2)^2 = a^2$$ ✓ calc magnitude

$$6\lambda^2 - 6\lambda + (17-a^2) = 0$$ ✓ simplify

For one solution to occur

$$\Delta = 0$$ ✓ discriminant

$$(-6)^2 - 4 \times 6 \times (17-a^2) = 0$$

$$a = \pm \frac{\sqrt{62}}{2}$$ ✓ solve

$$\therefore a = \frac{\sqrt{62}}{2} = 3.94 \text{ (2 d.p.)}$$

✓ positive

______End of Test______